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# ON INTEGRATION OF THE UNSTEADY EQUATION OF HEAT EXCHANGE BETWEEN BODIES MOVING IN FLUID

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The unsteady problem of convective heat exchange of bodies of a arbitrary shape moving in a perfect fluid or of drops moving in a viscous incompressible liquid, is considered in the approximation of thermal boundary layer.

The use of the concept of perfect fluid, i.e. of the small thickness of the viscous boundary layer in comparison with chracteristic dimensions of the body (or with the thickness of the thermal boundary layer at high Péclet numbers) in the first case is justified, for example, in the case of liquid metals used as heat carriers in atomic reactors [1]. It is linked with that the chracteristic Prandtl number $\Pr = \nu/\chi$ ($\nu$ and $\chi$ are coefficients of kinematic viscosity and thermal diffusivity) of liquid metals is contained in the interval $6\cdot 10^{-3}$ — $10^{-2}$ [1], and at high Péclet numbers $P = \Pr R = aU\chi^{-1}$ ($U$ is the characteristic velocity of the body) the respective Reynolds number $R$ is also high.

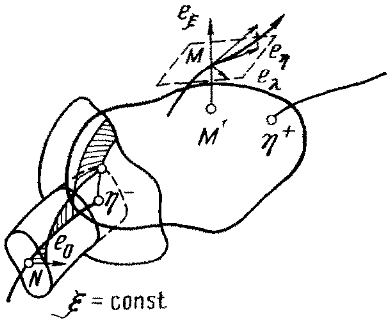


Fig. 1

The axisymmetric problem of unsteady convective diffusion to the absorbing body was considered in [2] in the case of steady flow, and the similar problem was investigated in [3] on the assumption that the stream function $\psi$ close to the body surface can be represented in the form of two factors each of which depends only on time or coordinate.

In the general case the expression for the stream function even in two-dimensional problems of heat exchange is not represented in the form of two coefficients (for instance, in the case of Stokes flow over a bubble, the superposition of an unsteady translational and of steady purely shear flows yields the expression $\psi \to (r-1)\sin^2\theta\,\{K(t) + {}^3/_2 \cdot \cos\theta\}$, $r \to 1$ [3], hence for solving this kind of problems it is necessary to have a more general method than proposed in [2,3].

Assuming that the fluid flow field has been determined by solving the respective problem of hydrodynamic flow, we introduce the local orthogonal system of coordinates $\xi$, $\eta$, $\lambda$ related to the body surface and the stream geometry, as was done in [3]. For this we determine the directions of unit vectors $\mathbf{e}_\xi$, $\mathbf{e}_\eta$, $\mathbf{e}_\lambda$ at some point $M$ near the particle. The nearest to $M$ point $M'$ of the body determines the direction of unit vector $\mathbf{e}_\xi$, and the segment $|MM'|$ defines the dimensionless coordinate $\xi$ normalized with respect to a characteristic dimension of the body. The direction of unit vector $\mathbf{e}_\eta$ is defined by that of the projection of the fluid velocity vector at point $M$ on the plane normal to $\mathbf{e}_\xi$, and the unit vector $\mathbf{e}_\lambda$ is selected so that the system of vectors $\mathbf{e}_\xi, \mathbf{e}_\eta, \mathbf{e}_\lambda$ forms an orthogonal trihedral (Fig. 1).

We shall call leading (trailing) stagnation point, the point at which a particular streamline reaches the body in whose neighborhood the normal velocity component of the fluid is directed toward (away from) the surface, and refer to the streamline emanating from that point as the flow-on (flow-off) trajectory. We assume that in the stagnation point neighborhood local smoothing of coordinate surfaces $\xi = \text{const}$ is provided and that all quantitites considered below have as many partial derivatives with respect to $\xi$ $(\xi \neq 0)$ as required. We draw through point $N$ of the flow-on trajectory the coordinate surface $\eta = \text{const}$, and in the tangent plane at point $N$ we fix vector $\mathbf{e}_0$. The direction of the flow-on or flow-off trajectories and of this vector determines the coordinate surface $\lambda = 0$ which is subsequently used as the reference plane. The parameter $\lambda$ is defined by the angle between $\mathbf{e}_0$ and the vector of the normal to the coordinate surface $\lambda = \text{const}$ at point $N$ $(0 \leqslant \lambda \leqslant 2\pi)$. The coordinate $\eta$ is determined by the length of arc of the intersection line of surface $\lambda = 0$ and of the particle surface $\xi = 0$ measured from point $N$ (Fig. 1).

In such coordinate system the fluid velocity vector is at every point of the form $\mathbf{v} = \{v_\xi, v_\eta, 0\}$ and has the following properties:

$$\xi \to 0,\ v_\xi = \xi O(1),\ v_\eta = v_\eta{}^\circ + \xi O(1);\ v_\eta{}^\circ = v_\eta\ (\xi = 0)$$

Note that unlike in the case of a steady flow field the directions of unit vectors $\mathbf{e}_\xi, \mathbf{e}_\eta, \mathbf{e}_\lambda$ and, also, the components of the metric tensor $g_{22}, g_{33}$ $(g_{11} \equiv 1)$ are time dependent.

The dimensionless equation of convective thermal conductivity in terms of the thermal boundary layer approximation in the system of coordinates $\xi, \eta, \lambda$ is of the form [3]

$$\frac{\partial T}{\partial t} - \frac{1}{\sqrt{g}} \frac{\partial (T, \Phi)}{\partial (\xi, \eta)} = P^{-1} \frac{\partial^2 T}{\partial \xi^2}, \quad g = g_{11} g_{22} g_{33} \tag{1}$$

where $\Phi = \Phi(\xi, \eta, \lambda, t)$ is the three-dimensional analog of the stream function [4] in which, in the case of steady hydrodynamics, a surface consisting entirely of streamlines corresponds to $\Phi = \text{const}$. Note that the coordinate $\lambda$ appears in Eq. (1) only as a parameter, hence it is henceforth omitted.

In proximity of the body surface function $\Phi$ can be represented, as $(\xi \to 0)$ [4], in the form

$$\Phi = \xi \Omega(t, \eta) \tag{2}$$

Initial and boundary conditions for Eq. (1), (2) are, so far, not specified, and will be defined later.

We introduce the variables

$$\omega = \sqrt{P}\, \xi f(t, \eta), \quad \zeta = \zeta(t, \eta) \tag{3}$$

where functions $f$ and $\zeta$ satisfy the following system of equations in partial derivatives:

$$\mathbf{L} f = \frac{1}{\sqrt{g}} \Omega_\eta' f, \quad \mathbf{L}(t, \eta) = \frac{\partial}{\partial t} + \frac{\Omega}{\sqrt{g}} \frac{\partial}{\partial \eta}, \quad \mathbf{L}\zeta = f^2 \tag{4}$$

Equation (1), (2) in variables (3), (4) reduces to the form

$$\partial T/\partial \zeta = \partial^2 T/\partial \omega^2 \tag{5}$$

and with the following initial and boundary conditions for temperature:

$$T(\omega, 0) = T_1(\omega);\ T(0, \zeta) = 0,\ T(\infty, \zeta) = 1 \tag{6}$$

has the solution

$$T(\omega, \zeta) = \int_0^\infty \frac{\sqrt{\omega\omega^*}}{2\zeta} \exp\left(-\frac{\omega^2 + \omega^{*2}}{4\zeta}\right) I_{1/2}\left(\frac{\omega\omega^*}{2\zeta}\right) T_1(\omega^*)\, d\omega^* \tag{7}$$

The boundary and initial conditions of the type (6) with $T_1(\omega) \not\equiv 1$ appear, for example, in problems of thermal (diffusion) interaction of several bodies in a fluid, when the concentration distribution in the heat trail region of the preceding body is defined by the expression $T_1(\omega)$ [4], and the concentration distribution in the diffusion boundary layer is determined by formula (7).

Let us now determine the expressions for the variables $\omega$ and $\zeta$. We integrate system (4) in variables $\eta$, $u(t, \eta)$, where $u(t, \eta) = C = \text{const}$ is the complete integral of the characteristic ordinary differential equation

$$dt = \sqrt{g}\, \Omega^{-1} d\eta \tag{8}$$

that corresponds to operator $\mathbf{L}$, i.e. $\mathbf{L}u = 0$.

For the unknown functions $f$ and $\zeta$ in variables $\eta$ and $u$ we have

$$\partial f/\partial\eta - [(\ln\Omega)_\eta' + (\ln\Omega)_u'\chi] f = 0$$
$$\partial\zeta/\partial\eta - \sqrt{g}\, \Omega^{-1} f^2 = 0 \tag{9}$$

where the expression in brackets is the partial derivative of $\ln\Omega(t, \eta)$ (expressed in variables $\eta$) with respect to $\eta, u$; and $\partial u/\partial\eta = \chi(\eta, u)$ defined in terms of the same variables.

Successive integration of Eqs. (9) yields the general formula for the sought functions $f$ and $\zeta$

$$f = A(u)\,\Omega E(\eta, u), \quad E(\eta, u) = \exp\left\{\int_{\eta_1}^{\eta} (\ln\Omega)_u'\chi\, d\eta^*\right\} \tag{10}$$

$$\zeta = A^2(u)\,[\Lambda(\eta, u) + B(u)], \quad \Lambda(a, b) = \int_{\eta^-}^{a} \sqrt{g}\,\Omega E^2(\eta^*, b)\, d\eta^*$$

where $\eta_1$ and $\eta^-$ are some fixed values of the variable $\eta$, and $A = A(u)$, $B = B(u)$ are arbitrary functions of the variable $u$ whose specific form is determined when the initial and boundary conditions are specified, as was done in [3].

In the self-similar case we have

$$T = \operatorname{erf}(\omega / 2\sqrt{\zeta}), \quad T_1(\omega) = 1 \tag{11}$$

The specific form of functions $A(u)$ is unimportant.

Representation (11) occurs, in particular, in the case of the following initial conditions:

$$T_\alpha(0, \xi, \eta) = 1, \quad T_\beta(0, \xi, \eta) = T_0(\xi, \eta) \tag{12}$$

of which the first cooresponds to the case when at $t < 0$ the stream temperature was constant and then at $t = 0$ heat exchange with the body surface saddenly begins, and the second to a steady heat exchange. Here and in what follows all quantities related to the first and second initial conditions (12) are denoted by subscripts $\alpha$ and $\beta$, respectively.

For the heat fluxes in the self-similar case we have

$$j(t, \eta, \lambda) = (g_{11}^{-1/2} \partial T / \partial \xi)_{\xi=0} = \sqrt{P/\pi} f \zeta^{-1/2} \tag{13}$$

$$I(t) = \iint_{\sigma} j (g^{1/2} g_{11}^{-1/2})_{\xi=0} \, d\eta \, d\lambda$$

where $\sigma$ is the surface of the body.

The variables of integration are defined as follows:

$$\zeta_\alpha = \Lambda(\eta, u) - \Lambda(u_0^{(-1)}(u), u), \quad \zeta_\beta = \zeta_\alpha + \zeta_0^* \tag{14}$$

$$f = \Omega E(\eta, u), \quad \zeta_0^* |_{t=0} = \zeta_0$$

$$\zeta_0^* = \zeta_0 (\eta = u_0^{(-1)}(u), u) = \int_{\eta^-}^{u_0^{-1}(u)} (\sqrt{g})_{\xi=0} \Omega(0, \eta^*) \, d\eta^*$$

where function $\eta = u_0^{(-1)}(u)$ is obtained by solving the equation $u_0 = u_0(\eta) = u(0, \eta)$ for $\eta$.

As an example, we consider the plane problem of heat exchange of a cylinder over which flows at variable velocity $U(t) = (1 + 2t)^{-1}$ a perfect incompressible fluid. In this case $\xi = r - 1, \ \eta = \pi - \theta, \ \sqrt{g} = 1$, and

$$\Omega(t, \theta) = 2(1 + 2t)^{-1} \sin \theta$$

For the variables $\zeta$ and $f$ we have ($u = \text{const}$ is the first integral of system (8))

$$\zeta_\alpha = 2u^{-1} [2 \operatorname{arctg} u + 2u(1 + u^2)^{-1} - \theta - \sin \theta]$$

$$\zeta_\beta = \zeta_\alpha + \zeta_0^*(u), \quad \zeta_0^*(u) = 4(1 + u^2)^{-1}$$

$$f = 2u^{-1} \sin \theta, \quad u = u(t, \theta) = \operatorname{tg}(\theta/2)(1 + 2t)$$

This shows that for considerable times the heat fluxes reach one and the same mode and approach zero at a rate proportional to $t^{-1/2}$, while the stream velocity tends to zero in proportion to $t^{-1}$.

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